Fast Rendering of Subdivision Surfaces

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Abstract

Subdivision surfaces provide a curved surface representation that is useful in a number of applications, including modeling surfaces of arbitrary topological type [5] , fitting scattered data [6] , and geometric compression and automatic level-of-detail generation using wavelets [8] . Subdivision surfaces also provide an attractive representation for fast rendering, since they can directly represent complex surfaces of arbitrary topology. This direct representation contrasts with traditional approaches such as trimmed NURBS, in which tesselating trim regions dominates rendering time, and algebraic implicit surfaces, in which rendering requires resultants, root finders, or other computationally expensive techniques.

We present a method for subdivision surface triangulation that is fast, uses minimum memory, and is simpler in structure than a naive rendering method based on direct subdivision. These features make the algorithm amenable to implementation on dedicated geometry engine processors, allowing high rendering performance on appropriately equipped graphics hardware.

CR Categories and Subject Descriptors: I.3.6 [Computer Graphics]: Methodology and Techniques.

Additional Key Words: subdivision surfaces, surface rendering.

1 Introduction

A subdivision surface results from iteratively refining a control mesh by adding new vertices into the mesh and perturbing the vertex locations using weighted averages of the vertex and its neighbors. The control mesh usually gives a rough polygonal approximation of the resulting surface; some methods even interpolate the control mesh[4]. The first subdivision surfaces were based on quadrilateral meshes and generalize biquadratic [3] and bicubic [1] tensor product surfaces.

We present an implementation of Loop's subdivision scheme [7] with additions by Hoppe *et al.* [6]. Loop's subdivision scheme is a generalization of C^2 quartic triangular B-splines, and is the simplest method known to lead to tangent plane smooth surfaces. The control mesh consists of triangular faces and, like subdivision surfaces in general, can have any topology. The resulting surface has the same topology as the control mesh.

Figure 1 diagrams Loop's subdivision process. Each iteration splits all the edges in the control mesh and introduces four new triangles for each original one. The location of a new vertex is obtained by a weighted average of the surrounding vertices. For example, when the edge BC is split, the vertex m is introduced at location $m = \frac{A+3B+3C+D}{8}$. Each iteration also perturbs the locations of existing vertices by taking a weighted average of the locations of the vertex and its immediate neighbors. Both of these operations—creating new vertices and perturbing the existing ones—can be expressed as matrix multiplications. Analysis of the eigenstructure of these matrices provides formulas for calculating the limiting location of each control vertex on the final surface. We can also compute two tangent vectors at each control vertex, leading directly to the normal vector at that point.

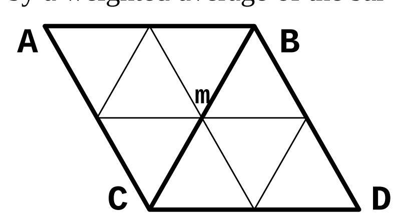


Figure 1: Triangle subdivision.

Hoppe *et al.* [6] extended Loop's scheme by introducing subdivision rules that lead to a piecewise smooth surface with features such as creases, corners, darts, and conical vertices. The vertices and edges of the control mesh are typed and the weights in the subdivision rule are chosen based on the vertex and edge types (the rules are collected in Appendix). Figure 2 shows a control mesh with color coded vertices and edges and the resulting subdivision surface.

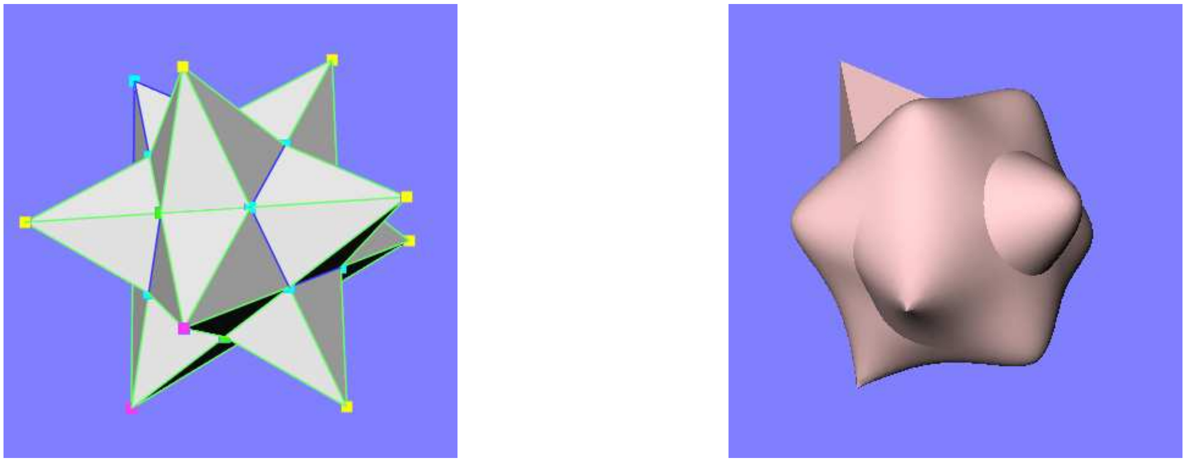


Figure 2 A control mesh and the resulting subdivision surface. Green edges are smooth while blue ones are sharp (creases). For vertices, yellow means smooth, magenta conical, green dart, and cyan a crease vertex (corner, regular, and irregular).

1.1 Rendering

Figure 1 suggests a simple method for rendering a subdivision surface: for each triangle, create four new triangles, and compute the locations of the resulting (original and new) vertices, then recurse. While this algorithm is straightforward, it is inappropriate for implementation in dedicated graphics hardware. Geometry engines (GEs) typically have limited memory and may be ill-suited to linked-list traversal and recursion.

We describe a novel algorithm for subdivision surface triangulation that is fast, uses little memory, and is easy to parallelize. These features make the method ideal for GE implementation. If GEs are available, then the host is freed from doing most of the work of generating triangles on the subdivision surface. In addition, instead of sending the GEs a long list of triangles, the host sends a compact representation of the surface, reducing host-GE bandwidth requirements. Even a machine without GEs, however, can reap the speed benefits of the simple indexing, iterative processing, and small memory consumption of our algorithm.

1.2 Overview of paper

In the next section we describe the full algorithm for rendering subdivision surfaces. We describe how to do the subdivision using a simple 2D array (Section 2.1.1) and present a sliding window method for rendering the subdivision surface patches depth-first and in-place, thus saving memory (Section 2.1.2). We then describe how we find the submeshes that are sent to GEs (Section 2.2) and describe some practical issues and limitations (Section 2.3). Section 3 presents our results and compares subdivision surface rendering to that of NURBS surface rendering. Section 4 discusses the promise for fast rendering when other subdivision surface features are considered such as automatic LOD selection and compression. An appendix describes the actual subdivision rules.

2 Method

Our basic idea is to divide the subdivision of an arbitrary control mesh into two tasks. The host manages the general control mesh and partitions it into small, regular patches. These patches can then be rendered independently inside a GE using a straightforward algorithm. The patches are stored so that after partitioning the host need only consult a cache and send the precomputed messages to the GEs to render more frames.

We have implemented the control mesh in the host as a half-edge data structure similar to ones presented in [10].

However, a much simpler data structure, a 2D array, suffices if the surface is rendered a pair of control mesh triangles at a time. The vertices can have arbitrary dimensions in either representation, so each vertex can have color or texture coordinates in addition to 3D space coordinates. The subdivision algorithm treats all dimensions in the same fashion.

In order to preserve continuity between patches, the patches need to overlap. Figure 3 shows two subdivisions of a patch consisting of a single control triangle. In the first subdivision, all the *support* vertices within the 1-neighborhood of the patch are needed both to split the edges and to update the (three) old vertices in the patch. The neighboring triangles are subdivided only enough to obtain a new layer of supporting vertices and edges. In the second subdivision this support layer shrinks even closer to the patch.

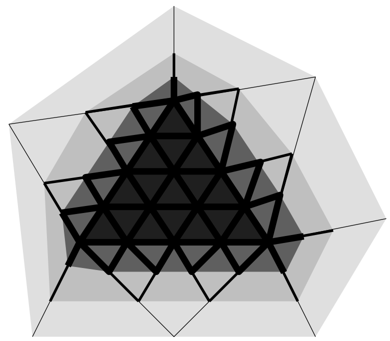


Figure 3 Neighbors are needed for subdivision.

We first describe how a 2D array can be used to subdivide pairs of control mesh triangles. Then we present a depth-first method for performing these calculations in a memory-efficient manner. We describe how we find the triangle pairs from the original control mesh, and finally describe the structure of the messages sent from host to GEs as well as some limitations caused by limited resources.

2.1 Subdivision inside a GE

2.1.1 Subdivision in a 2D array

A 2D array is perhaps the simplest possible data structure for describing points on a surface. However, it can store effectively only regularly tesselated surfaces of planar topology, so it is not a suitable representation for the general control mesh. Inside the control polygons things are much more regular: the surface has planar topology and the vertices are connected in a very regular fashion. The valence of each new vertex is always six, as can be seen from Figs. 1 and 4. When we combine two triangles into a quadrangle, we obtain a natural mapping from subdivided triangle pairs to a square array. Figure 4 shows the array representation of the triangle pair after one and two subdivisions.

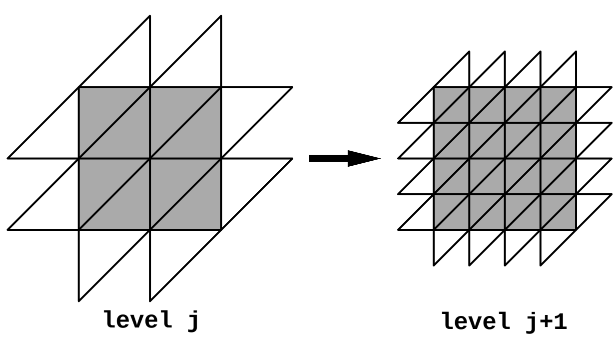


Figure 4 A 2D array stores vertices compactly.

There is no need to keep any connectivity information to find the neighbors for a vertex, and a vertexes' and edges' types can be obtained from their locations in the 2D array and from the types in the original message: corner vertices and the edges obtained by splitting original edges retain their type, all new edges are smooth, and all new vertices are smooth except when they lie on a crease edge, in which case they are regular crease vertices. Since there is no place for the extra corner support vertices in the 2D array, they are simply stored in an additional array.

With this scheme, it is easy to subdivide the triangles one level at a time. The size of the array at an intermediate level j is $(2^j + 3) \times (2^j + 3)$. At the final level we can calculate the vertex coordinates and normal vectors on the final surface, store them to a $(2^j + 1) \times (2^j + 1)$ matrix, and render them as triangle strips. An exception occurs if the (diagonal) edge between the original triangles is sharp. In that case all the vertices on that diagonal have two normal vectors, so we need one more element for each row for the extra normal, and each row must be rendered as two triangle strips.

2.1.2 The sliding window method

The simple method of subdividing a triangle pair one level at a time, even with a clever implementation, uses memory on the order of the size of the output. Since each level of subdivision quadruples the number of triangles, we could only implement a very modest number of levels (e.g., 2) in a GE. Instead, we use a depth-first approach that incrementally calculates the vertex coordinates, renders the triangles as soon as they are calculated, and reuses memory. This technique enables us to go several levels deeper than the naive approach for the same amount of memory.

It turns out that it is sufficient to store only a window of three rows of the 2D array for each level, plus two rows for the final vertex coordinates and normal vectors. In the course of the algorithm we slide the window down the 2D arrays. Figure 5 illustrates the initialization phase of the algorithm. On the left side we have the three first rows of the triangle array at subdivision level j. We obtain the first row support vertices at level $j + 1$ by splitting the vertical and

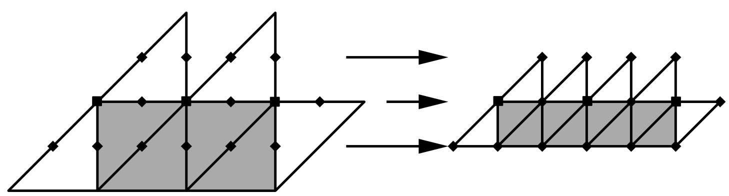


Figure 5 Initialization of the depth-first algorithm.

diagonal edges between the second and first row vertices at level j. Similarly, the second row is obtained by splitting the horizontal edges and updating the vertices of the second row at level j. The third row is obtained by splitting the vertical and diagonal edges connecting the second and third rows at level j. We initialize the three-row arrays a level at time, and at the final level we calculate the final coordinates and normal vectors on the top border (second row).

After the top rows have been initialized, we begin the recursive part of the algorithm. We introduce the next row of vertices from the original message. Figure 6 shows how a new row at level j provides the neighbors needed for splitting the horizontal edges and updating the vertex coordinates in the middle row. The new and updated vertices become the new low row at level $j + 1$, and they are written over the old high row at that level. We can now repeat the same going from level $j + 1$ to $j + 2$.

The new row at level j also enables us to split the diagonal and vertical edges connecting the level j low and middle rows, which produces yet another new row for level $j+1$. In this manner, we write new rows over old ones and descend towards the final subdivision level. There, with each new row, we calculate the final coordinates and normals for the middle row vertices and render another triangle strip using the new and the previous final values.

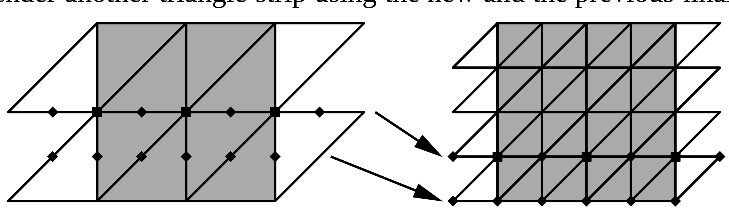


Figure 6 Introduction of the lowest row on the left enables us to calculate two new rows on the right.

Pseudocode for the depth-first algorithm looks like this:

```
SUBDIVIDE(level, last)
BEGIN
    IF (level != last) THEN
        split horizontal edges and update vertices
        SUBDIVIDE(level+1, last)
        split vertical and diagonal edges
        SUBDIVIDE(level+1, last)
    ELSE
        calculate final positions and normals
        render triangle strip
    ENDIF
END
```

Although the algorithm is not tail recursive, we can unroll the algorithm and execute it iteratively since we know the maximum number of subdivision levels.

Let us denote by n the number of 3D points (with normal vectors) that running the algorithm for l levels produces. The length of the rows at the finest level is approximately $\sqrt{n}$, and since the rows at a coarser level are only half as long as at the next finer level, the algorithm uses $O(\sum_{i=0}^{l} \sqrt{n}\frac{1}{2^i}) = O(\sqrt{n})$ memory (roughly $5 \cdot 2^{l+1}$ 3D vectors, $2 \cdot 2^{l+1}$ of which are used by the final vertices and normal vectors). Each vertex is calculated only once, the calculation takes constant time, and since there are only about one quarter as many vertices at a coarser level, the algorithm uses $O(\sum_{i=0}^{l} n\frac{1}{4^i}) = O(n)$ time.

2.2 Finding triangle pairs

The triangle pairs are found by a greedy algorithm that produces a pairing that seems nearly optimal. Each triangle is inserted into one of four sets (see Fig. 7) depending on how many free neighbors it has. Triangles with zero or one free neighbors have the highest priority (0), and the triangles with three free neighbors have the lowest priority (3). Of the remaining triangles (two free neighbors), the ones with both neighbors in set 3 have lower priority (2). The algorithm chooses a triangle from the highest nonempty set and pairs it with the neighbor of highest priority. The paired triangles are removed from the sets, and their remaining neighbors are moved to a higher priority set (from 2 and 1 to 0, from 3 to 2). Additionally, after moving a triangle from set 3 to 2, all the triangles in 2 with a neighbor in 0, 1, or 2 are moved to set 1.

set	free neighbors
0	0 or 1
1	2, one of them in 0 or 1
2	2, both of them in 3
3	3

Figure 7: The sets for pairing triangles.

Pairing a triangle with three free neighbors (set 3) may occur only once, in the beginning, if all the triangles form a closed surface. By removing the first triangle pair we create a hole in the mesh. Now we have some triangles along the boundary of the mesh with only one or two free neighbors. Pairing a triangle with only one free neighbor cannot take us away from the optimum solution: it could be that the neighbor was needed as a mate of some other triangle in the optimum pairing, but if we don't pair it, the triangle with one neighbor will be left alone.

The algorithm removes the "protruding" triangles at the boundary (in set 0), along with their only free neighbor, which usually makes some other triangle(s) have only one free neighbor. If we rid a triangle of *all* free neighbors, we can pair it with a dummy mate at any time without affecting the optimum solution; in such a case we also make a note so that the dummy mate won't be rendered twice. If we find that all the boundary triangles have only one boundary edge, we try to find two such neighboring triangles (set 1) and chip them off the mesh. In the rare case that we cannot find such pair, we randomly pair a border triangle (set 2) with a free neighbor, which again moves some triangles to set 0 or at least to set 1.

Figure 8 gives two examples how the algorithm proceeds. In the left mesh, all the triangles with a border edge belong to set 1, while the others belong to set 3. We start by pairing two triangles marked "1", which promotes one of triangles

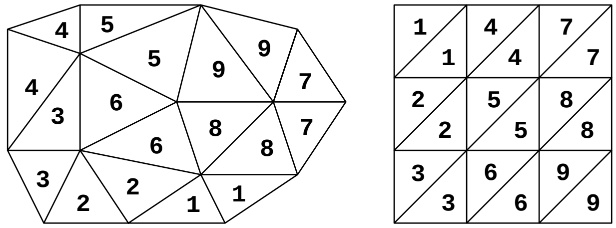


Figure 8 Pairing of triangles in a mesh.

marked “2” to set 1. They form the next pair, which now promotes one of the triangles marked “3” to set 0, and so forth. In the mesh on the right we always remove triangle pairs such that at least one of the triangles is left with only one free neighbor.

2.3 Some implementation issues

2.3.1 Host-GE interface

Figure 9 presents the information in the message that is sent to GEs for rendering two triangles. The message comprises two parts: a fixed-length part and a variable-length part. The fixed-length part contains the vertex coordinates along with vertex and edge types for the triangle pair (the shaded triangles) and the neighboring triangles (the four unshaded triangles). Additional information includes the dimension of vertex coordinates and whether the second triangle should be rendered or is a dummy. The variable-length part accommodates the variable valence of the control mesh vertices. We use a counter for each of the four corners followed by a list containing the position and type information for the corner support vertices and edges.

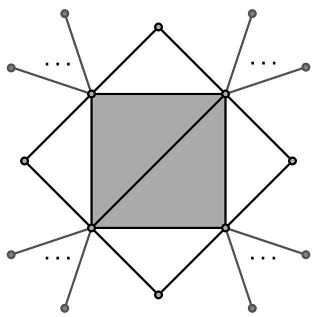


Figure 9 Two triangles and their immediate neighbors.

2.3.2 The overhead of subdividing the neighbors

When looking at Fig. 3 it might seem that we are doing more work subdividing the immediate neighbors than the actual patch. A closer look reveals, however, that after a few subdivisions the space and time spent on the support edges and vertices is but a small fraction of the real work. If we assume that we have a regular control mesh with vertices of valence six and a patch of two control triangles (as in Fig 4), the numbers of vertices and edges are as follows (i is the number of subdivisions):

	vertices		edges	
support	$4 \cdot 2^i + 6$	(38;72)	$8 \cdot 2^i + 6$	(72;134)
patch	$4^i + 2 \cdot 2^i + 1$	(81;289)	$3 \cdot 4^i + 2 \cdot 2^i$	(208;800)

The actual numbers at levels 3 and 4 are shown in parentheses.

A larger patch size with more control triangles would would mean less duplicated work with the support edges. However, a variable patch size would destroy the simple square array structure and the patch would not be guaranteed to be topologically a plane. Also, the patch size should be constant to obtain even load balancing for maximum performance in the case of several GEs.

2.3.3 Limitations

Due to the limited memory, there has to be a maximum number for the levels of subdivision we can process in a GE. If the user wants more subdivisions, the first levels can be done in the host, and the results are then sent to GEs.

The memory limitations also prevent us rendering meshes with arbitrarily large vertex valencies. Thus we have to impose a maximum limit for the corner support vertices for each triangle pair. We currently support a combined valency of 40 for the four corner vertices, though the choice is arbitrary and could be made somewhat larger. In the case the control mesh contains vertices of too large valency, the mesh would have to be modified by, e.g., introducing a small triangle at such a vertex (and three others to keep the mesh triangular) and distributing the valency among the new vertices.

3 Results

We implemented this algorithm as an experimental extension to OpenGL for the Silicon Graphics Onyx Reality Engine. The host stores the general control mesh using a half-edge data structure, chops the mesh into triangle pairs, and sends the pairs (with their neighbors) to GEs. The GE microcode (unoptimized C) subdivides the triangle pairs and renders them using the existing OpenGL code for rendering triangle strips.

3.1 Performance measurements

We present some performance measurements comparing subdivision surface and NURBS versions of the Utah teapot and a cylindrical car part with holes (see Fig. 10). In the following table, RE refers to an SGI Onyx RE2 with 12 geometry engines operating in parallel, while Indigo refers to an SGI Indigo^2 XL on which all subdivision work is done in the host.

Subdivision surfaces				
model	control points	triangles	triangles/sec (1000s)	
			RE	Indigo
teapot	157	4960	147	26
		19840	207	31
car part	126	3264	157	26
		13056	212	39
NURBS surfaces				
teapot	512	8740	141	24
car part	1143+189	5608	80	22

All subdivision surface times include finding triangle pairs. The additional control points for the “car part” NURBS surface indicate trim curve points; measured times include trim region tesselation. The two sets of subdivision surface timing numbers are for 2 and 3 levels of subdivision.

The results show that even our unoptimized implementation was faster than the OpenGL NURBS, both for a GE microcode implementation (RE) and when the subdivision was performed completely on the host (Indigo). In addition, for a fixed number of triangles, the visual quality of a subdivision surface rendering is generally better than that obtained with a NURBS rendering. The reason is that the subdivision surface representation concentrates more triangles in areas of higher detail; this effect is difficult to achieve with NURBS renderers.

The triangle rate increases with the number of subdivision levels. There seem to be three reasons: with more subdivision levels, the setup and communication costs are amortized over a larger number of triangles, more of the computed

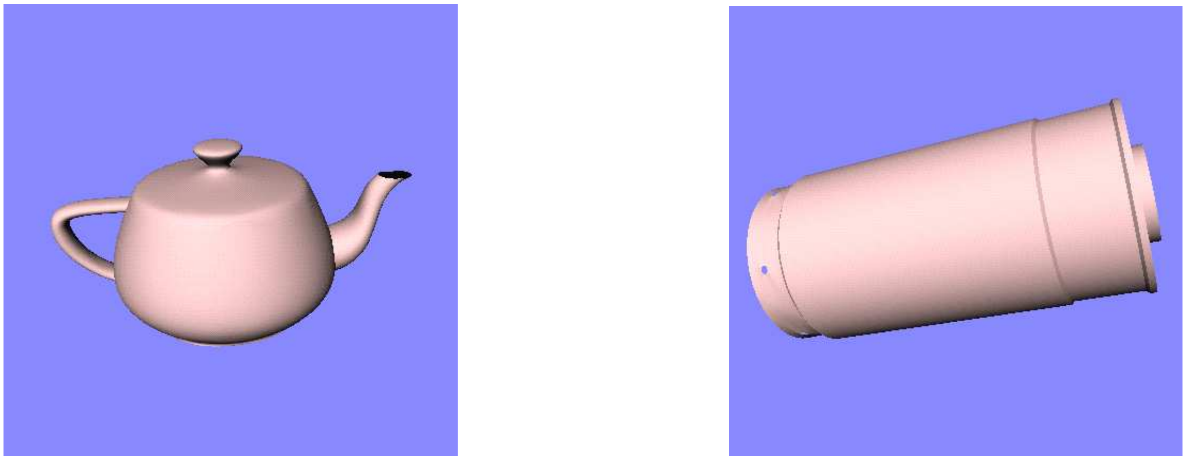


Figure 10 Subdivision surfaces: the Utah teapot and a cylindrical car part with holes.

triangles are inside the control triangles where the subdivision algorithm is most regular, and individual triangle strips become longer.

3.2 Real time editing

We implemented a simple surface editor that allows us to move the control vertices around and change the types of the control vertices and the connecting edges. The fast rendering allows interactive editing of the control mesh even for quite complicated objects (consisting of hundreds of control triangles) and gives real-time feedback by showing the resulting subdivision surface. Figure 11 shows a model of a head and the result of a couple of minutes' worth of editing.

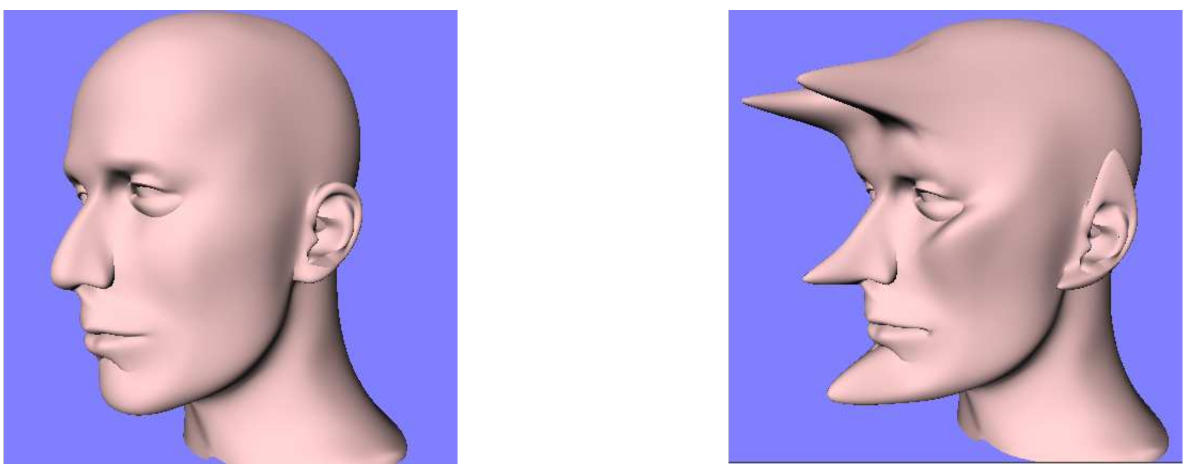
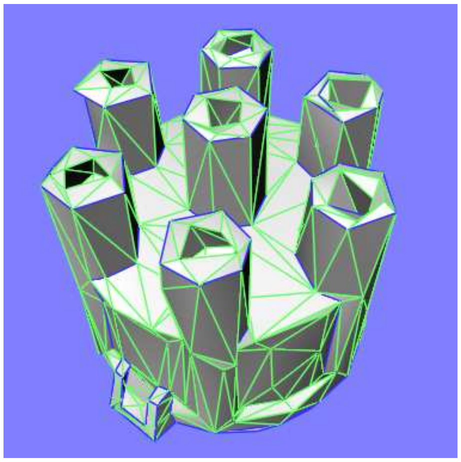


Figure 11 A mannequin head before and after editing.

3.3 Comparison with NURBS

Subdivision surfaces possess several potential advantages over NURBS patches. The most obvious advantage is that a single subdivision surface can model an object of any topology, whereas objects with complicated shapes usually need to be covered by several NURBS patches. Figure 12 shows such a surface. With a single continuous surface there is no need to stitch patches together to avoid cracks or to trim patches to avoid surface interpenetration. Further, the control points of a NURBS patch need to be in a topologically regular 2D lattice, whereas the subdivision surface control mesh gives more freedom about the connectivity of the control points.



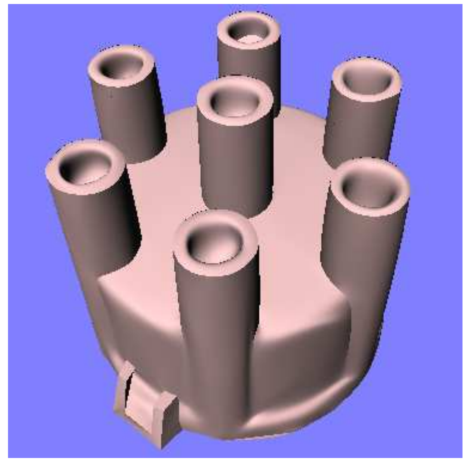


Figure 12 A distributor cap: control mesh and the resulting surface.

4 Discussion

4.1 Compression in communication

Rendering a (piecewise) smooth surface accurately by polygonal approximation requires a large number of polygons. Our method not only relieves the host from the task of tesselating the surface into a set of triangles, but it also reduces communication between host and GEs. Let's assume that the user wants to subdivide the control mesh four times to achieve a rather high geometrical accuracy. Every triangle in the control mesh is subdivided four times, yielding $4^4 = 256$ triangles. When these triangles are rendered, for example as triangle strips, we still have to send each vertex twice (on average) to the graphics hardware. A single subdivision message, however, sends typically only 10-15 vertices (two triangles and their neighbors), along with some type information, but produces 512 triangles.

Subdivision surfaces combined with wavelet analysis may yield even greater compression in communication [2]. A multiresolution representation of the control mesh would allow sending larger submeshes compactly.

4.2 Level-of-detail control

In order to obtain interactive rendering speeds, automatic level-of-detail (LOD) control is essential. With subdivision surfaces, there is no need to store or create alternate descriptions of geometry at different levels of detail. The LOD control is built in: one parameter, the number of subdivision levels, dictates how detailed the rendering is.

It is quite easy to obtain a continuous LOD control by interpolating between subdivision levels. This is important for eliminating "jumping" of the surface, e.g., when one zooms in to an object to display more detail. Suppose we want to display the object at subdivision level 3.5. Let us call the vertices introduced at level j even and vertices at level $j+1$ odd. We would then calculate the surface points at level 4, use the average of the parent vertices (even) for each odd vertex as the starting point, and the actual location as the ending point. The non-integer subdivision levels between 3 and 4 can now be obtained by interpolating between the starting and ending points.

It is also possible to render different control mesh triangle pairs at different levels of subdivision. A naive implementation would leave cracks between patches, but with a bit of extra information, the cracks can easily be filled. Let's assume that while rendering the current patch we know that the patch on the left was rendered with one fewer levels of subdivision. We know that each leftmost vertex of the even rows corresponds to a vertex on the neighboring patch. Introducing a new triangle consisting of the even, odd, and the next even vertex will fill the crack. In general, if there are more levels between the patches, the crack filling polygons consist of the vertices shared by the patches and of all the finer level vertices between them.

4.3 Conclusions

We have presented an algorithm for rendering Loop-Hoppe subdivision surfaces that uses very little memory and is efficient to execute. A control mesh of arbitrary topology is efficiently divided into patches consisting of triangle pairs and their immediate neighbors. The patches can be rendered separately and they are about the same size, which makes the algorithm attractive for parallel implementation if several GEs are available.

Subdivision surfaces have several advantages over traditional surface descriptions, such as NURBS. Arbitrarily complicated objects can be represented by a single surface rather than a collection of patches; the control mesh is quite natural to edit by designers and provides both smooth surfaces and sharp features; level-of-detail control is supported in a natural way; and our implementation outperformed the OpenGL NURBS versions of comparable surface models.

Acknowledgements

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Appendix

The edges and vertices in the control mesh can have the following types. An edge can be either smooth (continuous surface normals) are a crease (discontinuous surface normals). A vertex can be either smooth, dart, crease, or corner, depending whether it has zero, one, two, or more incident crease edges, respectively. A crease vertex is regular iff there is exactly two smooth edges between the two creases, otherwise it is nonregular. Additionally, any vertex can be tagged as a conical vertex.

Figure 13 presents the masks used for both subdividing the control mesh and calculating the final positions and normal vectors at each vertex location. A new value is obtained by an affine combination of a vertex and its neighbors, and the mask gives the appropriate weights. For example, a new vertex position can be obtained by $v_0' = \sum_{i=0}^{n} c_i v_i / \sum_{i=0}^{n} c_i$, where v_0 is the old vertex position and the other v_i's are the neighbors.

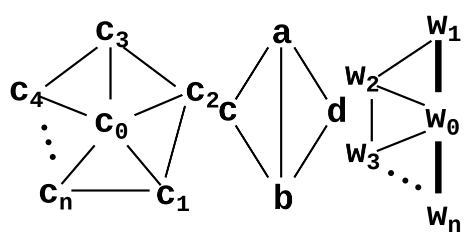

Figure 13 The subdivision masks for vertices and edges.

There are several cases[1] for applying the leftmost vertex mask both for subdivision (S) and calculating the final position (F):

[1]Formulas from [6], except for conical vertices from [9].

	vertex type	c_0	c_i
S	smooth or dart	$n/a(n) - n$	1
S	crease	6	edge c_0-c_i crease ? 1 : 0
S	corner	1	0
S	conical	b(n)	1
F	smooth or dart	3n/(8a(n))	1
F	regular crease	4	edge c_0-c_i crease ? 1 : 0
F	nonregular crease	3	edge c_0-c_i crease ? 1 : 0

where $a(n) = 5/8 - (3 + 2\cos(2\pi/n))^2/64$ and $b(n) = (3 + 2\cos(2\pi/n)/8$.

The center mask is used when subdividing edges (between a and b). If the edge is smooth or if either of the vertices (a or b) is a dart, the weights {a,b,c,d} are {3,3,1,1}. If the edge is a crease and one of the edge vertices (a) is a regular crease while the other (b) is either non-regular crease or corner, the weights are {5,3,0,0}. In the remaining cases the weights are {1,1,0,0}.

The normal vectors can be calculated by taking the cross product of two surface tangent vectors u_1 and u_2. For non-crease vertices, we use the left mask. The center weight c_0 is 0 for both u_1 and u_2, while $c_i = \cos(2\pi i/n)$ for u_1 and $c_i = \cos(2\pi(i-1)/n)$ for u_2. For crease vertices we use the right mask (the fat line denotes the crease), where u_1 goes along and u_2 across the crease. The u_1 weights are $w_1 = 1, w_n = -1$, other w_i's are 0. For a regular crease vertex, u_2 weights $\{w_0, ..., w_4\}$ are {-2,-1,2,2,-1}. For a non-regular crease vertex, u_2 weights are, for $n \geq 4$, $w_0 = 0$, $w_1 = w_n = \sin\theta$, and $w_i = (2\cos\theta - 2)(\sin(i-1)\theta)$ for $1 < i < n$, where $\theta = \pi/(n-1)$; for $n = 3$, $\{w_0, ..., w_3\}$ are {-1,0,1,0}; for $n = 2$, $\{w_0, w_1, w_2\}$ are {-2,1,1}.

References

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